Division of the momentum of electromagnetic waves in linear media into electromagnetic and material parts

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It is proposed a natural and consistent division of the momentum of electromagnetic waves in linear, non-dispersive and non-absorptive dielectric and magnetic media into material and electromagnetic parts. The material part is calculated using directly the Lorentz force law and the electromagnetic momentum density has the form $\varepsilon_0\mathbf{E}\times\mathbf{B}$, without an explicit dependence on the properties of the media. The consistency of the treatment is verified through the obtention of a correct momentum balance equation in many examples and showing the compatibility of the division with the Einstein's theory of relativity by the use of a *gedanken* experiment. An experimental prediction for the radiation pressure on mirrors immersed in linear dielectric and magnetic media is also made.

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I. INTRODUCTION

There has been an extensive debate about the correct expression for the momentum density of electromagnetic waves in linear media. The Minkowski's expression $\mathbf{D}\times\mathbf{B}$ and the Abraham's $\mathbf{E}\times\mathbf{H}/c^2$ are the most famous ones, both proposed in the beginning of the twentieth century [1]. In these expressions, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{D}\equiv\varepsilon_0\mathbf{E}+\mathbf{P}$ is the electric displacement field and $\mathbf{H}$ is defined as $\mathbf{H}\equiv\mathbf{B}/\mu_0-\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the electric and magnetic dipole densities of the medium. ε_0 is the permittivity of free space, μ_0 is the permeability of free space and $c=1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in vacuum. These expressions make qualitatively different predictions for the momentum of light in a medium. The Minkowski formulation predicts that a photon with momentum $\hbar\omega/c$ in vacuum increases its momentum to $n\hbar\omega/c$, where n is the refraction index of the medium, on entering a dielectric medium. The Abraham expression, however, states that the photon momentum decreases to $\hbar\omega/nc$ on entering the medium.

The debate of which form is the correct one persisted for many decades, with experiments and theoretical discussions, from time to time, seeming to favor either one of the two formulations. Arguments that sometimes are naively used in favor of the Minkowski formulation are the experiments of Jones *et al.* [2, 3] that measured the radiation pressure on mirrors immersed in dielectric media, the experiments of Ashkin and Dziedzic [4] that measured the radiation pressure on the free surface of a liquid dielectric, the experiments of Gibson *et al.* [5] that measured the radiation pressure on the charges of a semiconductor via the photon drag effect and the experiments of Campbell *et al.* [6] that measured the recoil momentum of atoms in a gas after absorbing one photon. All these experiments are consistent with the consideration that a photon in a medium has momentum $n\hbar\omega/c$. Arguments that sometimes are naively used in favor of the Abraham formulation are the symmetry of its energy-momentum tensor, compatible with conservation of angular momentum, the agreement of its electromagnetic momentum density with the predictions of Einstein box theories [7–9] in a direct way, the experiments of Walker *et al.* [10, 11] that measured the torque on a dielectric disk suspended in a torsion fiber subjected to external magnetic and electric fields and the experiments of She *et al.* [12] that measured a push force on the end face of a silica filament when a light pulse exits it. These experiments are consistent with the Abraham form for the momentum density of an electromagnetic wave in a dielectric medium.

Although the debate is still supported by some researchers, Penfield and Haus showed, more than forty years ago, that neither of the forms is completely correct on its own [13]. The electromagnetic momentum of electromagnetic waves in linear media is always accompanied by a material momentum, and when the material counterpart is taken into account, both the Minkowski and Abraham forms for the electromagnetic momentum density yield the same experimental predictions. They are simply two different ways, among many others, to divide the total momentum density. A revision of this discussion and the eventual conclusion can be found in Ref. [1]. In fact, the experimental results of Jones *et al.* were reproduced by Gordon [14] and Loudon [15] using the Abraham form for the electromagnetic momentum density and calculating the material momentum by means of the Lorentz force. Gordon also reproduced the results of Ashkin and Dziedzic by the same way [14]. Loudon *et al.* [16] showed that the experiments of Gibson *et al.* can also be explained with both formulations. And Leonhardt [17] showed that the experiments of Campbell *et al.* can also be explained by the use of the Abraham form for the momentum density and a redefinition of the mechanical momentum. On the other side, Israel [18] has derived the experimental results of Walker *et al.* using

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the Minkowski formulation with a suitable combination of electromagnetic and material energy-momentum tensors and the conclusions of the experiments of She *et al.* were recently questioned [19–21].

What happens is that for each experimental situation one formulation can predict the behavior of the system in a simpler way, but the Minkowski, Abraham and other proposed formulations are always equivalent. In a recent article, Pfeifer *et al.* show the conditions in which we can neglect the material counterpart of the Minkowski energy-momentum tensor [22], justifying why it is possible to predict the behavior of the experiments of Jones *et al.* and the modeling of optical tweezers only with the electromagnetic tensor in the Minkowski formulation. To summarize, I quote Ref. [1]: "(...) no electromagnetic wave energy-momentum tensor is complete on its own. When the appropriate accompanying energy-momentum tensor for the material medium is also considered, experimental predictions of all various proposed tensors will always be the same, and the preferred form is therefore effectively a matter of personal choice."

But we can ask if there is, among all possible ways to divide the total momentum density of an electromagnetic wave in a medium into electromagnetic and material parts, a natural one. I believe there is. $\mathbf{E}$ and $\mathbf{B}$ are the fields that appear in the Lorentz force law and actually interact with electric charges. They are the fields that can transfer momentum to matter. So, in my point of view, they must be the fields that may carry electromagnetic momentum. $\mathbf{D}$ and $\mathbf{H}$ should be seen as averaged quantities of material and electromagnetic properties, used to simplify the calculations. In this sense, it seems natural that the electromagnetic part of the momentum density of an electromagnetic wave in a medium has the form $\mathbf{p}_{\mathrm{e.m.}} = \varepsilon_0 \mathbf{E}\times\mathbf{B}$, which does not have an explicit dependence on the properties of the medium. The material part of the momentum should be calculated as the momentum acquired by the medium by the action of the Lorentz force on the charges of the medium. An electromagnetic energy-momentum tensor that has these characteristics was previously proposed by Obukhov and Hehl [23]. Here I show the validity of this division of the momentum density in a series of examples. The form $\varepsilon_0\mathbf{E}\times\mathbf{B}$ for the electromagnetic momentum density is equivalent to the Abraham one in non-magnetic media. As in all cited experiments the media under consideration were non-magnetic, there is no difference between the treatment with this momentum density and the Abraham one. Only in magnetic media the differences will appear.

Gordon [14], Loudon [15, 24] and Mansuripur [25] calculated the material momentum density of electromagnetic waves in linear non-dispersive dielectrics using directly the Lorentz force law. Scalora *et al.* [26] used numerical simulations to calculate, also by the Lorentz force law, the momentum transfer to more general dispersive media. Recently, Hinds and Barnett [27] used the Lorentz force to calculate the momentum transfer to a single atom. Here, following this method in an analytical treatment, I show that it may exist permanent transfers of momentum to the media due to the passage of electromagnetic pulses that were not considered before. I also generalize the previous treatments considering magnetic media. Mansuripur has treated magnetic materials in another work [28], but the force equation, results and conclusions of this work are different from his. I believe that my treatment is more adequate. If this method of using the Lorentz force is adopted to calculate the material momentum of electromagnetic waves in linear media, we conclude that the electromagnetic part of the momentum density must have the form $\varepsilon_0\mathbf{E}\times\mathbf{B}$ in order that we have momentum conservation in the various circumstances that are discussed in this paper.

In Sec. II, I calculate the material momentum density of an electromagnetic pulse in a homogeneous linear dielectric and magnetic medium. In Sec. III, I calculate the momentum transfer to the medium near the interface on the partial reflection and transmission of an electromagnetic pulse by the interface between two linear media and show the momentum conservation in the process. In Sec. IV, I show the compatibility of the present treatment with the experiments of Jones *et al.* [2, 3] for the momentum transfer from an electromagnetic wave in a dielectric medium to a mirror upon reflection, show the momentum conservation in this process and generalize the treatment of Mansuripur [29] for the radiation pressure on mirrors immersed in linear media for arbitrary kinds of mirrors and magnetic media. In Sec. V, I use my method to calculate the momentum transfer to an antireflection layer between two linear media on the passage of an electromagnetic pulse and show the momentum conservation in the process. In Sec. VI, I show the compatibility of the proposed division of the momentum density with the Einstein's theory of relativity by the use of a *gedanken* experiment of the kind "Einstein box theories". Finally, in Sec. VII, I present my concluding remarks.

II. MATERIAL MOMENTUM OF ELECTROMAGNETIC WAVES IN LINEAR DIELECTRIC AND MAGNETIC MEDIA

The momentum transfer to a linear non-absorptive and non-dispersive dielectric medium due to the presence of an electromagnetic wave can be calculated directly using the Lorentz force [14, 15, 24–26]. The force acting on electric dipoles can be written as

$$\mathbf{F}_{\mathrm{dip.}} = (\mathbf{p}\cdot\nabla)\mathbf{E} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\times\mathbf{B}\;, \tag{1}$$

where $\mathbf{p}$ is the dipole moment. In a linear, isotropic and non-dispersive dielectric, the electric dipole moment density can be written as $\mathbf{P} = \chi_{\mathrm{e}}\varepsilon_0\mathbf{E}$, where χ_{e} is the electric susceptibility of the medium. It is important to stress that the consideration of a non-dispersive linear medium

must be seen as an approximation, once every material medium is inevitably accompanied of dispersion. But if the electromagnetic wave has a narrow frequency spectrum and the dispersion is small in this region of frequencies, this treatment will give the material contribution of the total momentum of the wave with a good precision. Using Eq. (1), the Maxwell equations and some vectorial identities, the force density on this medium can be written as [14]

$$\mathbf{f}_{\text{diel.}} = \chi_{\text{e}}\varepsilon_0 \left[\nabla\left(\frac{1}{2}E^2\right) + \frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right)\right] . \tag{2}$$

In magnetized media, there is also a bound current $\nabla\times\mathbf{M}$ that is affected by the Lorentz force. So the force density in a linear non-dispersive dielectric and magnetic medium can be written as

$$\mathbf{f} = \underbrace{\chi_{\text{e}}\varepsilon_0\nabla\left(\frac{1}{2}E^2\right)}_{\mathbf{f}_1} + \underbrace{\chi_{\text{e}}\varepsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\times\mathbf{B}\right)}_{\mathbf{f}_2} + \underbrace{\left(\nabla\times\mathbf{M}\right)\times\mathbf{B}}_{\mathbf{f}_3} . \tag{3}$$

I will calculate the material momentum due to the action of the forces $\mathbf{f}_1$, $\mathbf{f}_2$ and $\mathbf{f}_3$ separately. In a linear, isotropic and non-dispersive magnetic medium, we have $\mathbf{M} = \chi_{\text{m}}\mathbf{H} = \chi_{\text{m}}/[(1+\chi_{\text{m}})\mu_0]\mathbf{B}$, where χ_{m} is the magnetic susceptibility of the medium.

In his treatment of the material part of the momentum of electromagnetic waves in magnetic materials [28], Mansuripur uses a specific model for a magnetic medium, obtaining a different equation for the bound currents in the medium. The form $\nabla\times\mathbf{M}$ is more general and agrees with his form in a homogeneous medium. As the interfaces between different linear media will be treated here, the general form for the bound currents must be used. He also takes the vector product of the bound currents with $\mu_0\mathbf{H}$ instead of $\mathbf{B}$ to find the force density. I don't think this is adequate. For these reasons, I believe that the present treatment to find the material momentum of electromagnetic waves in magnetic media is more adequate than that of Mansuripur.

Consider an electromagnetic plane wave propagating in $\hat{\mathbf{z}}$ direction described by the following electric field:

$$\mathbf{E}_{\text{i}}(z,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}\omega \tilde{E}(\omega)\mathrm{e}^{i\left(\frac{n\omega}{c}z-\omega t\right)}\hat{\mathbf{x}} , \tag{4}$$

with $\tilde{E}(-\omega) = \tilde{E}^*(\omega)$ and $\mathbf{B}_{\text{i}} = (n/c)|\mathbf{E}_{\text{i}}|\hat{\mathbf{y}}$, $n = \sqrt{(1+\chi_{\text{e}})(1+\chi_{\text{m}})}$ being the refraction index of the medium. The consideration of a plane wave pulse must also be seen as an approximation. We can consider a beam with a small angular spread, in which all wavevectors that compose it are very close to the z axis such that their z component are equal to their norm in a good approximation. For the wave of Eq. (4), the force densities $\mathbf{f}_1$ and $\mathbf{f}_3$ from Eq. (3) can be written as

$$\mathbf{f}_1 = -\frac{\chi_{\text{e}}\varepsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) , \tag{5}$$

$$\mathbf{f}_3 = \frac{\chi_{\text{m}}(1+\chi_{\text{e}})\varepsilon_0}{2}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B}) . \tag{6}$$

Substituting these expressions for $\mathbf{f}_1$ and $\mathbf{f}_3$, the material momentum density of an electromagnetic wave in a homogeneous linear non-dispersive and non-absorptive medium can be written as

$$\mathbf{p}_{\text{mat}}(t) = \int_{-\infty}^{t} \mathrm{d}t'\mathbf{f}(t') = \frac{(\chi_{\text{e}}+\chi_{\text{m}}+\chi_{\text{e}}\chi_{\text{m}})}{2}\varepsilon_0\mathbf{E}(t)\times\mathbf{B}(t). \tag{7}$$

It is important to stress that this material momentum density propagates with the pulse and disappears after its passage through the medium. In the remaining part of this work, it will be considered that the total momentum density of an electromagnetic wave in a homogeneous linear medium is given by the sum of the material momentum density above and the electromagnetic momentum density $\varepsilon_0\mathbf{E}\times\mathbf{B}$. By also calculating the permanent transfers of momentum to the media by the action of the Lorentz force in some situations, the momentum conservation in these processes and the consistency of the proposed division of the momentum density will be shown.

III. REFLECTION AND TRANSMISSION OF AN ELECTROMAGNETIC PULSE BY THE INTERFACE BETWEEN TWO LINEAR MEDIA

Consider the situation illustrated in Fig. 1. Initially we have a pulse of electromagnetic radiation in medium 1, with electric susceptibility $\chi_{\text{e}1}$, magnetic susceptibility $\chi_{\text{m}1}$ and refraction index $n_1 = \sqrt{(1+\chi_{\text{e}1})(1+\chi_{\text{m}1})}$, propagating in $\hat{\mathbf{z}}$ direction towards the interface with medium 2, with electric and magnetic susceptibilities $\chi_{\text{e}2}$ and $\chi_{\text{m}2}$ and refraction index $n_2 = \sqrt{(1+\chi_{\text{e}2})(1+\chi_{\text{m}2})}$. The interface is in the plane $z = 0$ and the incidence is normal. Later, we will have a transmitted pulse in medium 2 and a reflected pulse in medium 1. It will be shown that the proposed division of the momentum density of the pulse in electromagnetic and material parts leads to the momentum conservation in the process. Representing the electric field of the incident pulse as in Eq. (4), the electromagnetic part of its momentum can be written as

$$\mathbf{P}_0 = \int \mathrm{d}^3r\varepsilon_0\mathbf{E}_{\text{i}}\times\mathbf{B}_{\text{i}} = \varepsilon_0\int\mathrm{d}x\int\mathrm{d}y\int_{-\infty}^{+\infty}\mathrm{d}\omega|\tilde{E}(\omega)|^2\hat{\mathbf{z}} . \tag{8}$$

The integrals in x and y are, in principle, infinite. But a plane wave is always an approximation, so the amplitude $\tilde{E}(\omega)$ must decay for large x and y. I will not worry about that, only assume that the integral is finite. Integrating also the material momentum density of Eq. (7), we find that the total momentum of the incident ($\mathbf{P}_{\text{i}}$), reflected

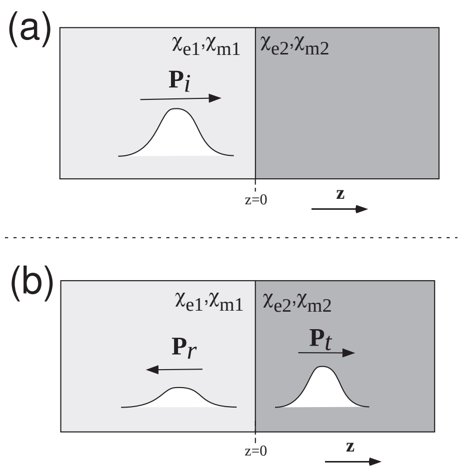

FIG. 1: (a) A pulse with total momentum $\mathbf{P}_\mathrm{i}$ propagates in medium 1 with electric and magnetic susceptibilities $\chi_{\mathrm{e}1}$ and $\chi_{\mathrm{m}1}$ towards the interface with medium 2 with electric and magnetic susceptibilities $\chi_{\mathrm{e}2}$ and $\chi_{\mathrm{m}2}$. (b) The resultant reflected and transmitted pulses with total momentum $\mathbf{P}_\mathrm{r}$ and $\mathbf{P}_\mathrm{t}$.

($\mathbf{P}_\mathrm{r}$) and transmitted ($\mathbf{P}_\mathrm{t}$) pulses are

$$\mathbf{P}_\mathrm{i} = \left(1 + \frac{\chi_{\mathrm{e}1} + \chi_{\mathrm{m}1} + \chi_{\mathrm{e}1}\chi_{\mathrm{m}1}}{2}\right)\mathbf{P}_0 \ , \ \mathbf{P}_\mathrm{r} = -|r|^2\mathbf{P}_\mathrm{i} \ ,$$
$$\mathbf{P}_\mathrm{t} = |t|^2\left(1 + \frac{\chi_{\mathrm{e}2} + \chi_{\mathrm{m}2} + \chi_{\mathrm{e}2}\chi_{\mathrm{m}2}}{2}\right)\mathbf{P}_0 \ , \qquad (9)$$

where r and t are the reflection and transmission coefficients, respectively.

There is also a momentum transfer to medium 1 during reflection that does not propagate with the electromagnetic pulse. We can observe in Eq. (3) that it is the total (incident plus reflected) field that generates the force density $\mathbf{f}_1$. As $E^2 = E_\mathrm{i}^2 + E_\mathrm{r}^2 + 2\mathbf{E}_\mathrm{i}\cdot\mathbf{E}_\mathrm{r}$ in medium 1, we must consider the term $\mathbf{f}_1' = \chi_{\mathrm{e}1}\varepsilon_0\nabla(\mathbf{E}_\mathrm{i}\cdot\mathbf{E}_\mathrm{r})$. The momentum transfer due to this term is

$$\begin{aligned}\mathbf{P}_1' &= \int_{-\infty}^{+\infty}\mathrm{d}t\int\mathrm{d}x\int\mathrm{d}y\int_{-\infty}^{0}\mathrm{d}z\chi_{\mathrm{e}1}\varepsilon_0\frac{\partial}{\partial z}(\mathbf{E}_\mathrm{i}\cdot\mathbf{E}_\mathrm{r})\,\hat{\mathbf{z}}\\ &= r\chi_{\mathrm{e}1}\mathbf{P}_0 \ ,\end{aligned} \qquad (10)$$

with $\mathbf{P}_0$ given by Eq. (8).

As $\mathbf{E}_\mathrm{i}\times\mathbf{B}_\mathrm{r}+\mathbf{E}_\mathrm{r}\times\mathbf{B}_\mathrm{i} = 0$, the force density $\mathbf{f}_2$ in Eq. (3) does not contribute to a permanent momentum transfer to the medium. The permanent momentum transfer from Eq. (10) was not considered in the previous treatments of reflection of electromagnetic pulses by interfaces between two dielectrics [15, 24, 25], so these works are compatible with momentum conservation only when the incidence medium is vacuum and $\mathbf{P}_1' = 0$.

The force density $\mathbf{f}_3$ in Eq. (3) also contributes to a permanent transfer of momentum to medium 1. We can see that for the pulse of Eq. (4) this force density can be written as

$$\mathbf{f}_3 = -\frac{\chi_\mathrm{m}}{\mu_0(1+\chi_\mathrm{m})}\nabla\left(\frac{B^2}{2}\right) \ . \qquad (11)$$

Repeating the calculation of Eq. (10), remembering that $\mathbf{B}_\mathrm{r} = -r\mathbf{B}_\mathrm{i}$, we can see that this force density transfers a momentum $\mathbf{P}_3'$ to the medium 1 given by

$$\mathbf{P}_3' = r\chi_{\mathrm{m}1}(1+\chi_{\mathrm{e}1})\mathbf{P}_0 \ . \qquad (12)$$

There is still another contribution to the momentum transfer to the interface due to the discontinuity of $\mathbf{M}$ in the interface. In an extent δz much smaller than the wavelength of light around $z = 0$, $\mathbf{f}_3$ from Eq. (3) can be written as

$$\mathbf{f}_3|_{z=0} = \frac{1}{\delta z}\left[\frac{\chi_{\mathrm{m}2}}{1+\chi_{\mathrm{m}2}}B_2 - \frac{\chi_{\mathrm{m}1}}{1+\chi_{\mathrm{m}1}}B_1\right]\frac{(-\hat{\mathbf{x}})}{\mu_0}\times\mathbf{B} \ ,$$

where $\mathbf{B}_1 = \mathbf{B}_\mathrm{i}(1-r)$ is the magnetic field just before the interface, in medium 1, and $\mathbf{B}_2 = [(1+\chi_{\mathrm{m}2})/(1+\chi_{\mathrm{m}1})]\mathbf{B}_1$ is the magnetic field just after the interface, in medium 2. Integrating in this region of extent δz and in x, y and t, we obtain the following momentum transfer to this interface:

$$\begin{aligned}\mathbf{P}_4' &= \int_{-\infty}^{+\infty}\mathrm{d}t\int\mathrm{d}x\int\mathrm{d}y\int_{-\delta z/2}^{+\delta z/2}\mathrm{d}z\ \mathbf{f}_3 \qquad (13)\\ &= \frac{(\chi_{\mathrm{m}1}-\chi_{\mathrm{m}2})(1+\chi_{\mathrm{e}1})(1-r)^2}{2}\left[1+\frac{1+\chi_{\mathrm{m}2}}{1+\chi_{\mathrm{m}1}}\right]\mathbf{P}_0.\end{aligned}$$

Using Eqs. (9), (10), (12) and (13) and substituting the values of r and t [30],

$$\begin{aligned}r &= \frac{\sqrt{(1+\chi_{\mathrm{e}1})(1+\chi_{\mathrm{m}2})} - \sqrt{(1+\chi_{\mathrm{e}2})(1+\chi_{\mathrm{m}1})}}{\sqrt{(1+\chi_{\mathrm{e}1})(1+\chi_{\mathrm{m}2})} + \sqrt{(1+\chi_{\mathrm{e}2})(1+\chi_{\mathrm{m}1})}} \ ,\\ t &= \frac{2\sqrt{(1+\chi_{\mathrm{e}1})(1+\chi_{\mathrm{m}2})}}{\sqrt{(1+\chi_{\mathrm{e}1})(1+\chi_{\mathrm{m}2})} + \sqrt{(1+\chi_{\mathrm{e}2})(1+\chi_{\mathrm{m}1})}} \ ,\end{aligned}$$

it can be shown that

$$\mathbf{P}_\mathrm{r} + \mathbf{P}_\mathrm{t} + \mathbf{P}_1' + \mathbf{P}_3' + \mathbf{P}_4' = \mathbf{P}_\mathrm{i} \ . \qquad (14)$$

The obtention of a correct momentum balance equation shows the consistency of the proposed division of the momentum of the wave into material and electromagnetic parts.

IV. RADIATION PRESSURE ON SUBMERGED MIRRORS

In the second example, consider that we put a very good conductor in place of medium 2 in Fig. 1 that reflects an electromagnetic pulse described by Eq. (4). The momentum transfer to this mirror can be calculated using the Lorentz force that acts on the induced currents

in the mirror. The magnetic field inside the mirror ($z > 0$) can be written as [30]

$$\mathbf{B}_{\rm mir} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \frac{2n_1}{(1+\chi_{\rm m1})c} \tilde{E}(\omega) e^{(-\kappa + ik)z - i\omega t} \hat{\mathbf{y}} , \tag{15}$$

where k and κ are real functions of ω. This form for the magnetic field guarantees the continuity of $\mathbf{H}$ at the interface and decrease exponentially with z. Disregarding the small presence of the electric field inside the mirror that would generate absorption of electromagnetic energy, the current density in the mirror can be written as $\mathbf{J}_{\rm mir} = (1/\mu_0)\nabla \times \mathbf{B}$. So the momentum transfer to the mirror due to the reflection of the electromagnetic pulse is

$$\begin{aligned}\mathbf{P}_{\rm mir} &= \int dx \int dy \int_0^{+\infty} dz \int_{-\infty}^{+\infty} dt\, \mathbf{J}_{\rm mir} \times \mathbf{B}_{\rm mir} \quad (16)\\ &= \int dx \int dy \int_{-\infty}^{+\infty} d\omega \frac{2\varepsilon_0(1+\chi_{\rm e1})}{(1+\chi_{\rm m1})} |\tilde{E}(\omega)|^2 \frac{(\kappa - ik)}{\kappa} \hat{\mathbf{z}} .\end{aligned}$$

We have $|\tilde{E}(-\omega)| = |\tilde{E}(\omega)|$, $\kappa(-\omega) = \kappa(\omega)$ and $k(-\omega) = -k(\omega)$, such that the integral in ω of the term that multiplies $(-ik)$ is zero. So the momentum transfer from the pulse to the mirror upon reflection is

$$\mathbf{P}_{\rm mir} = 2\left(\frac{1+\chi_{\rm e1}}{1+\chi_{\rm m1}}\right) \mathbf{P}_0 , \tag{17}$$

with P_0 given by Eq. (8).

The same momentum transfer is obtained with the condition $\mathbf{P}_{\rm mir} = \mathbf{P}_{\rm i} - \mathbf{P}_{\rm r} - \mathbf{P}'_1 - \mathbf{P}'_3 - \mathbf{P}'_4$ using Eqs. (9), (10), (12) and (13) with $r = -1$ and $\chi_{\rm m2} = 0$. On this way, we arrive at the same expression (17) for $\mathbf{P}_{\rm mir}$, showing again the consistency of the treatment.

For non-dispersive linear media, the total energy density of the wave can be written as $u_{\rm tot} = (1+\chi_{\rm e1})\varepsilon_0|\mathbf{E}|^2$ [30]. So the energy of the incident pulse of Eq. (4) is

$$U_{\rm i} = \int d^3r (1+\chi_{\rm e1})\varepsilon_0 E_{\rm i}^2 = \frac{(1+\chi_{\rm e1})c}{n_1} |\mathbf{P}_0|. \tag{18}$$

The ratio between the modulus of the momentum transfer to the mirror (17) and the incident energy (18) is $2n_1/[(1+\chi_{\rm m1})c]$, in accordance with the experiments of Jones *et al.* [2, 3]. In these experiments, the magnetic susceptibilities of the media were too small to affect the results, so the ratio is usually stated as $2n_1/c$. These experiments are frequently used to support the Minkowski formulation, but we can see that the present treatment also predicts the measured results. As this treatment is for non-dispersive media, it does not say whether the term c/n_1 in this expression is the group velocity or the phase velocity of the wave in the medium. The experiments show that it is the phase velocity [3].

In a recent paper [29], Mansuripur treated the problem of radiation pressure on mirrors immersed in linear dielectric and non-magnetic media using a model of a medium with imaginary refraction index to describe the mirrors. He considered the case where the complex reflection coefficient of the mirror is $e^{i\phi}$ and calculated the Lorentz force on the electric currents of the mirror. In the present treatment of the problem of the reflection of an electromagnetic pulse by a non-magnetic mirror with this complex reflection coefficient immersed in a linear dielectric and magnetic medium, the momentum transfer to the mirror can be calculated using Eqs. (9), (10), (12) and (13) as $\mathbf{P}_{\rm mir} = \mathbf{P}_{\rm i} - \mathbf{P}_{\rm r} - \mathbf{P}'_1 - \mathbf{P}'_3 - \mathbf{P}'_4$ with $r = e^{i\phi}$ and $\chi_{\rm m2} = 0$. Retaining terms up to the first power in $\chi_{\rm m1}$, we obtain

$$\mathbf{P}_{\rm mir} \simeq 2\left[1 + (\chi_{\rm e1} - \chi_{\rm m1} - \chi_{\rm e1}\chi_{\rm m1}) \sin^2\left(\frac{\phi}{2}\right)\right] \mathbf{P}_0 ,$$

which is compatible with the result reported by Mansuripur when $\chi_{\rm m1} = 0$ and can be experimentally tested. As I rely only on the properties of the linear medium and no particular model to describe the mirror is adopted, this treatment is more general than that of Mansuripur.

V. MOMENTUM TRANSFER TO AN ANTIREFLECTION LAYER

In the next example, suppose that we have an antireflection layer between media 1 and 2 consisted of a material with electric susceptibility $\chi'_{\rm e}$ obeying $(1+\chi'_{\rm e}) = \sqrt{(1+\chi_{\rm e1})(1+\chi_{\rm e2})}$, with magnetic susceptibility $\chi'_{\rm m}$ obeying $(1+\chi'_{\rm m}) = \sqrt{(1+\chi_{\rm m1})(1+\chi_{\rm m2})}$ and thickness $\lambda'/4$, λ' being the wavelength of the central frequency of the pulse in this medium. If we have an almost monochromatic incident pulse in medium 1 propagating towards the interface, it will be almost completely transmitted to medium 2. The initial and final situations are illustrated in Fig. 2. If the electric field of the incident pulse $\mathbf{E}_{\rm i}$ is described by Eq. (4), the electric field of the transmitted one will be [31]

$$\begin{aligned}\mathbf{E}_2(z,t) &= \left[\frac{(1+\chi_{\rm e1})(1+\chi_{\rm m2})}{(1+\chi_{\rm e2})(1+\chi_{\rm m1})}\right]^{1/4} \times \\ &\quad \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \tilde{E}(\omega) e^{i\left(\frac{n_2\omega}{c}z - \omega t\right)} \hat{\mathbf{x}} . \quad (19)\end{aligned}$$

The total momentum of the incident pulse $\mathbf{P}_{\rm i}$ and of the transmitted pulse $\mathbf{P}_2$ can be written as

$$\begin{aligned}\mathbf{P}_{\rm i} &= \left(1 + \frac{\chi_{\rm e1} + \chi_{\rm m1} + \chi_{\rm e1}\chi_{\rm m1}}{2}\right) \mathbf{P}_0 , \quad (20)\\ \mathbf{P}_2 &= \left[\frac{(1+\chi_{\rm e1})(1+\chi_{\rm m2})}{(1+\chi_{\rm e2})(1+\chi_{\rm m1})}\right]^{1/2} \times \\ &\quad \times \left(1 + \frac{\chi_{\rm e2} + \chi_{\rm m2} + \chi_{\rm e2}\chi_{\rm m2}}{2}\right) \mathbf{P}_0 , \quad (21)\end{aligned}$$

with $\mathbf{P}_0$ given by Eq. (8). Since the momentum of the initial and final pulses are different, there must be a momentum transfer to the antireflection layer in order that

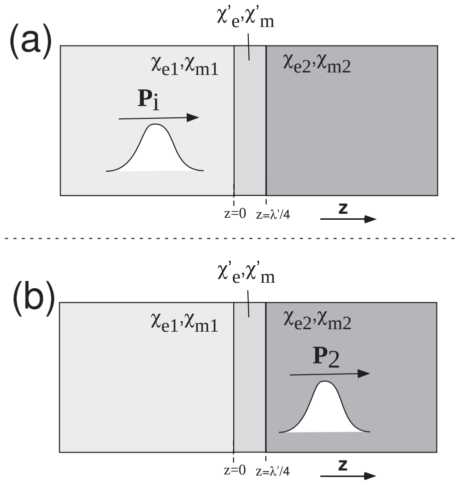

FIG. 2: (a) A pulse with total momentum $\mathbf{P}_\mathrm{i}$ propagates in medium 1 with electric and magnetic susceptibilities χ_{e1} and χ_{m1} towards medium 2 with electric and magnetic susceptibilities χ_{e2} and χ_{m2}. There is an antireflection coating layer between media 1 and 2 consisted of a material with electric and magnetic susceptibilities χ'_e and χ'_m such that $(1+\chi'_\mathrm{e}) = \sqrt{(1+\chi_{\mathrm{e1}})(1+\chi_{\mathrm{e2}})}$ and $(1+\chi'_\mathrm{m}) = \sqrt{(1+\chi_{\mathrm{m1}})(1+\chi_{\mathrm{m2}})}$. The thickness of the layer is $\lambda'/4$, λ' being the wavelength of the central frequency of the pulse in this medium. Despite the figure, it is assumed that the pulse is much larger than the layer. (b) The pulse was totally transmitted to medium 2 and has total momentum $\mathbf{P}_2$.

we have momentum conservation. Now I will show that the use of the force density of Eq. (3) acting in the antireflection layer guarantees momentum conservation in the process. Let us call $\mathbf{E}'$ and $\mathbf{B}'$ the electric and magnetic fields in the region of the layer. The boundary conditions impose

$$\begin{gathered}\mathbf{E}'|_{z=0} = \mathbf{E}_1|_{z=0}\ ,\ \mathbf{E}'|_{z=\lambda'/4} = \mathbf{E}_2|_{z=\lambda'/4}\ ,\\ \frac{\mathbf{B}'|_{z=0}}{1+\chi'_\mathrm{m}} = \frac{\mathbf{B}_1|_{z=0}}{1+\chi_{\mathrm{m1}}}\ ,\ \frac{\mathbf{B}'|_{z=\lambda'/4}}{1+\chi'_\mathrm{m}} = \frac{\mathbf{B}_2|_{z=\lambda'/4}}{1+\chi_{\mathrm{m2}}}\ .\end{gathered} \tag{22}$$

Writing $\mathbf{f}_3$ from Eq. (3) as in Eq. (11) and integrating the force density $\mathbf{f}$ on time and in the volume of the layer, we obtain the momentum transfer to the layer:

$$\begin{aligned}\mathbf{P}''_\mathrm{a} &= \int \mathrm{d}x \int \mathrm{d}y \int_0^{\lambda'/4} \mathrm{d}z \int_{-\infty}^{+\infty} \mathrm{d}t\ \mathbf{f} \\ &= \frac{\chi'_\mathrm{e}+\chi'_\mathrm{m}+\chi'_\mathrm{e}\chi'_\mathrm{m}}{2}\left[\sqrt{\frac{(1+\chi_{\mathrm{e1}})(1+\chi_{\mathrm{m2}})}{(1+\chi_{\mathrm{e2}})(1+\chi_{\mathrm{m1}})}}-1\right]\mathbf{P}_0.\end{aligned} \tag{23}$$

We must also consider the momentum transfers to the interfaces between the layer and the mediums 1 and 2 due the discontinuities of the magnetization $\mathbf{M}$. Calling $\mathbf{P}''_\mathrm{b}$ the momentum transfer in $z=0$ and $\mathbf{P}''_\mathrm{c}$ the momentum transfer in $z=\lambda'/4$ and repeating the treatment of Sec. III, we obtain

$$\mathbf{P}''_\mathrm{b} = \frac{(\chi_{\mathrm{m1}}-\chi'_m)(2+\chi_{\mathrm{m1}}+\chi'_m)(1+\chi_{\mathrm{e1}})}{2(1+\chi_{\mathrm{m1}})}\mathbf{P}_0, \tag{24}$$

$$\begin{aligned}\mathbf{P}''_\mathrm{c} &= \frac{(\chi'_m-\chi_{\mathrm{m2}})(2+\chi'_m+\chi_{\mathrm{m2}})(1+\chi_{\mathrm{e2}})}{2(1+\chi_{\mathrm{m2}})}\times \\ &\times\sqrt{\frac{(1+\chi_{\mathrm{e1}})(1+\chi_{\mathrm{m2}})}{(1+\chi_{\mathrm{e2}})(1+\chi_{\mathrm{m1}})}}\mathbf{P}_0\ .\end{aligned} \tag{25}$$

The total momentum transfer to the antireflection layer due to the passage of the pulse is $\mathbf{P}'' = \mathbf{P}''_\mathrm{a} + \mathbf{P}''_\mathrm{b} + \mathbf{P}''_\mathrm{c}$. Using Eqs. (20), (21), (23), (24) and (25), we can see that

$$\mathbf{P}_2 + \mathbf{P}'' = \mathbf{P}_\mathrm{i} \tag{26}$$

and momentum is conserved in the process. Again, we see the consistency of the proposed division of the momentum of the wave.

VI. EINSTEIN BOX THEORIES

As a last example, we can see whether the present treatment agrees with Einstein's theory of relativity testing a *gedanken* experiment of the kind "Einstein box theories" in which a single photon in free space with energy $\hbar\omega$ and momentum $\mathbf{P}_0 = (\hbar\omega/c)\hat{\mathbf{z}}$ is transmitted through a transparent slab with electric and magnetic susceptibilities χ_{e2} and χ_{m2}, refraction index $n_2 = \sqrt{(1+\chi_{\mathrm{e2}})(1+\chi_{\mathrm{m2}})}$, length L, mass M and antireflection layers in both sides. Due to propagation in the medium, the photon suffers a spacial delay $(n_2-1)L$ in relation to propagation in vacuum. According to the theory of relativity, the velocity of the center of mass-energy of the system must not change due to the passage of the photon through the slab, so the slab must suffer a displacement Δz such that [7–9]

$$Mc^2\Delta z = \hbar\omega(n_2-1)L\ . \tag{27}$$

The use of the Abraham momentum density as the electromagnetic part of the total momentum density gives directly the correct displacement of the slab, so this *gedanken* experiment is frequently used to support the Abraham formulation. Now I will show that the present treatment also gives the correct displacement in a direct way. The mechanical momentum of the slab $\mathbf{P}_{\mathrm{slab}}$ during the passage of the photon has 2 contributions. The first is the momentum transferred to the first antireflection layer. The second is the material part of the momentum of the photon in the slab. Using Eqs. (7), (21), (23),(24) and (25) with $\chi_{\mathrm{m1}} = \chi_{\mathrm{e1}} = 0$, we find that the total momentum of the slab is

$$\mathbf{P}_{\mathrm{slab}} = \frac{n_2-1-\chi_{\mathrm{m2}}}{n_2}\frac{\hbar\omega}{c}\hat{\mathbf{z}}\ . \tag{28}$$

After the passage of the pulse, the momentum of the slab comes back to zero. Part of this momentum is in

the form of hidden momentum [13, 32, 33], a relativistic effect that is not associated with the movement of the slab. A magnetic dipole $\mathbf{m}$ in an uniform electric field $\mathbf{E}_0$ has a hidden momentum $\mathbf{m} \times \mathbf{E}_0$. So the density of hidden momentum of an electromagnetic wave in a linear medium is given by $\mathbf{M}\times\mathbf{E}$. Integrating this density in volume, the total hidden momentum of the pulse in the slab is given by $\mathbf{P}_{\mathrm{hid}} = -\chi_{\mathrm{m}2}\hbar\omega/(n_2 c)\hat{\mathbf{z}}$. To find the velocity of the slab, we must subtract the hidden momentum from its total momentum and divide by its mass. As the pulse takes a time $n_2 L/c$ to pass through the slab, the displacement of the slab is

$$\Delta z = \frac{|\mathbf{P}_{\mathrm{slab}} - \mathbf{P}_{\mathrm{hid}}|}{M}\frac{n_2 L}{c} = \frac{\hbar\omega(n_2-1)L}{Mc^2} \,,$$

in agreement with Eq. (27). One more time we see the consistency of the proposed division of the momentum of the wave into electromagnetic and material parts.

VII. CONCLUSIONS

In summary, it was shown that the momentum density of electromagnetic waves in linear non-absorptive and non-dispersive dielectric and magnetic media can be naturally and consistently divided into an electromagnetic part $\varepsilon_0\mathbf{E}\times\mathbf{B}$, which has the same form independently of the medium, and a material part that can be obtained directly using the Lorentz force. This division was shown to be consistent with momentum conservation in various circumstances and with the "Einstein box theories". I believe that it may be possible to extend this division to all kinds of media.

I also calculated permanent transfers of momentum to the media due to the passage of electromagnetic pulses that were missing in previous treatments [15, 24, 25], showed the compatibility of the division with the experiments of Jones *et al.* [2, 3] and generalized the treatment of radiation pressure on submerged mirrors [29], which can be submitted to experimental verification.

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[1] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Rev. Mod. Phys. **79**, 1197 (2007).
[2] R. V. Jones and J. C. S. Richards, Proc. R. Soc. London A **221**, 480 (1954).
[3] R. V. Jones and B. Leslie, Proc. R. Soc. London A **360**, 347 (1978).
[4] A. Ashkin and J. M. Dziedzic, Phys. Rev. Lett. **30**, 139 (1973).
[5] A. F. Gibson, M. F. Kimmitt, A. O. Koohian, D. E. Evans, and G. F. D. Levy, Proc. R. Soc. Lond. A **370**, 303 (1980).
[6] G. K. Campbell, A. E. Leanhardt, J. Mun, M. Boyd, E. W. Streed, W. Ketterle, and D. E. Pritchard, Phys. Rev. Lett. **94**, 170403 (2005).
[7] O. R. Frisch, Contemp. Phys. **7**, 45 (1965).
[8] I. Brevik, Eur. J. Phys. **2**, 37 (1981).
[9] R. Loudon, Fortschr. Phys. **52**, 1134 (2004).
[10] G. B. Walker, D. G. Lahoz, and G. Walker, Can. J. Phys. **53**, 2577 (1975).
[11] G. B. Walker and G. Walker, Can. J. Phys. **55**, 2121 (1977).
[12] W. She, J. Yu, and R. Feng, Phys. Rev. Lett. **101**, 243601 (2008).
[13] P. Penfield Jr. and H. A. Haus, *Electrodynamics of moving media* (MIT, Cambridge, 1967).
[14] J. P. Gordon, Phys. Rev. A **8**, 14 (1973).
[15] R. Loudon, J. Mod. Opt. **49**, 821 (2002).
[16] R. Loudon, S. M. Barnett, and C. Baxter, Phys. Rev. A **71**, 063802 (2005).
[17] U. Leonhardt, Phys. Rev. A **73**, 032108 (2006).
[18] W. Israel, Phys. Lett. **67B**, 125 (1977).
[19] M. Mansuripur, Phys. Rev. Lett. **103**, 019301 (2009).
[20] M. Mansuripur and A. R. Zakharian, Phys. Rev. A **80**, 023823 (2009).
[21] I. Brevik, Phys. Rev. Lett. **103**, 219301 (2009).
[22] R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, and H. Rubinsztein-Dunlop, Phys. Rev. A **79**, 023813 (2009).
[23] Y. N. Obukhov and F. W. Hehl, Phys. Lett. A **311**, 277 (2003).
[24] R. Loudon, Phys. Rev. A **68**, 013806 (2003).
[25] M. Mansuripur, Opt. Express **12**, 5375 (2004).
[26] M. Scalora, G. D'Aguanno, N. Mattiucci, M. J. Bloemer, M. Centini, C. Sibilia, and J. W. Haus, Phys. Rev. E **73**, 056604 (2006).
[27] E. A. Hinds and S. M. Barnett, Phys. Rev. Lett. **102**, 050403 (2009).
[28] M. Mansuripur, Opt. Express **15**, 13502 (2007).
[29] M. Mansuripur, Opt. Express **15**, 2677 (2007).
[30] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1999), 3rd ed.
[31] M. Born and E. Wolf, *Principles of Optics* (Macmillan, New York, 1964), 2nd ed.
[32] W. Schockley and R. P. James, Phys. Rev. Lett. **18**, 876 (1967).
[33] L. Vaidman, Am. J. Phys. **58**, 978 (1990).